IOPscience iopscience.iop.org

Home Search Collections Journals About Contact us My IOPscience

Anomaly on a submanifold system-new index theorem related to a submanifold system

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1995 J. Phys. A: Math. Gen. 28 1399

(http://iopscience.iop.org/0305-4470/28/5/024)

View the table of contents for this issue, or go to the journal homepage for more

Download details:
IP Address: 171.66.16.68
The article was downloaded on 02/06/2010 at 01:13

Please note that terms and conditions apply.

J. Phys. A: Math. Gen. 28 (1995) 1399–1412. Printed in the UK

Anomaly on a submanifold system—new index theorem related to a submanifold system

Shigeki Matsutani
2-4-11 Sairenji, Niihama, Ehime 792, Japan

Received 27 June 1994

Abstract. Recently the submanifold quantum system has been studied. In this article, after we confine a Dirac field in a thin curved rod, we find an anomalous term in the fermionic field theory related to the extrinsic curvature. In other words, we find a new Atiyah–Singer-type index theorem related to the geometry of the submanifold. As the anomaly is associated with the nonlinear Schrödinger equation as a loop soliton, we also discuss it.

1. Introduction

The quantum submanifold system is studied in material science (da Costa 1981, Matsutani 1993a). It was found by Jensen and Koppe (1971) and formulated by da Costa (1981) for a *general shape of the submanifold in three-dimensional space* $\mathbb{R}^3$. *After quantizing a particle* over $\mathbb{R}^3$, we confine a particle in a submanifold in terms of a confinement potential. Its properties are different from those of the manifold quantum system, which is well known in elementary particle and gravitational physics (Birrell and Davies 1982). In submanifold quantum mechanics the extrinsic curvature plays an important role, though in manifold (without boundary) quantum mechanics it does not.

Furthermore it also differs from quantum mechanics on a submanifold in terms of the Dirac constraint scheme with a certain condition. If a surface in $\mathbb{R}^3$ is expressed by $f = 0$, one can impose $f = 0$ as the Dirac constraint and formally quantize a particle over the surface (Ikegami *et al* 1992). However, the $f = 0$ condition seems to be incompatible with the principle of differential geometry and kinematic theory; in these theories, the equation is given locally and, after integration, we obtain global information, e.g. the manifold. This result does not, therefore, agree with that from submanifold quantum mechanics to which we have already referred. However, there is another choice; we can make a particle satisfy the condition $\dot{f} = 0$ in terms of the Dirac constraint scheme. This is within the framework of local theory and its result is in agreement with submanifold quantum mechanics if we select Weyl ordering (Ikegami *et al* 1992, Matsutani 1993b). Thus submanifold quantum mechanics appears to be well defined.

However, a further question arises. If the differential operator defined over the submanifold is a natural object, it should carry the geometrical information. Does the differential equation given through the submanifold quantum scheme reflect the symmetry?

Mathematically the question means 'does the Atiyah–Singer-type index theorem exist for the submanifold system?'. The Atiyah–Singer index theorem unifies the analysis and the geometry in mathematics (Atiyha and Singer 1968a, b, Gilkey 1984, Nakahara 1990). There is a well defined analogy between them, in the meaning of category theory, and both give the same invariant or an integer. Physically this is related to the anomaly that is a kind of twist in the functional space of fermionic quantum fields; the integration of the anomaly over the base space generates the index theorem. However, it is for a manifold with and without boundary including a fibre bundle. Though in the theorem for the manifold with boundary we come across the characteristic terms consisting of an extrinsic curvature along the boundary, it could not be extended to a submanifold system in general; e.g. the dimension of a boundary must be equal to the dimension of the manifold minus one (for example, Branson and Gilkey 1992). We require a canonical differential operator for the submanifold system and a new index theorem for the submanifold system. However, the requirement has been partially satisfied by the author who found a new Atiyah–Singer-type index theorem for a rod on two-dimensional space $\mathbb{R}^2$ in terms of the Dirac operator which is naturally defined in the submanifold quantum scheme (Matsutani 1994a). Physically we discovered a new anomaly in the fermionic field theory related to the submanifold system. Our main purpose in this article is to generalize the anomaly of the rod in $\mathbb{R}^2$ to one in $\mathbb{R}^3$. The generalization gives the direction for a rod on $\mathbb{R}^n$ (Matsutani 1994b).

In general, the geometric part of an anomaly is related to an integrable equation (Fujikawa 1979, Nakahara 1990). In our anomaly, the geometrical object is regarded as a loop soliton if the fermionic system satisfies a certain condition. By studying the submanifold quantum system, the physical meaning of the quantum mechanics of soliton physics was revealed by Tsuru and the author (Matsutani and Tsuru 1992). While a thin elastic rod on $\mathbb{R}^2$ is governed by the modified Korteweg–de Vries (MKdV) equation, the Dirac operator in the rod agrees with the Lax operator of the MKdV equation. When the fermionic system and the base rod system are adiabatically coupled, the energy spectrum of the fermion does not depend on the time development of the base space and it represents the infinite conserved quantities of the motion of the rod. In other words, the fictitious quantum mechanics in soliton physics for the MKdV equation is realized as the Dirac fermionic system confined in a thin elastic rod (Matsutani and Tsuru 1992, Matsutani 1994a, b).

On the other hand, an elastic rod in $\mathbb{R}^3$ obeys the nonlinear Schrödinger equation (NLSE) and it is known as the Hashimoto vortex soliton (HVS) (Hashimoto 1972, Tsuru 1987). The author applied our theory to the Dirac system on an elastic rod in $\mathbb{R}^3$ (Matsutani 1994b). Then we found that the Dirac equation is in agreement with Lax's eigenvalue equation of the NLSE. Accordingly the generalization that we will perform appears to have geometrical meaning.

Recently the shape effect on a real material has been studied. Tokihiro and Hanamura (1993) considered the linear response of excitons in a curved polymer. Alternatively, a large polymer, e.g. DNA, sometimes behaves like an elastica in $\mathbb{R}^3$ that is governed by the NLSE (Tsuru and Wadati 1986, Tsuru 1987). Furthermore the topology of DNA on the configuration space is very important; one is concerned about whether it is knotted or unknotted or whether it is twisted or not (Schlick and Olson 1992, Shaw and Wang 1993). In this article, we will not deal with the knot of a rod directly but introduce an index related to the geometry of the submanifold, the summation of the signed crossings of the curve. Thus it is of physical interest to investigate the relation between the fermion on a curved rod and the configuration of the curve as a model of the anomaly between the electron and the DNA.

In this article, we will generalize the anomaly for a Dirac field in a thin rod on $\mathbb{R}^2$ to one on $\mathbb{R}^3$ and investigate its geometrical meaning. In the derivation of the anomaly we will use Fujikawa's procedure (Fujikawa 1979) because it is a natural strategy.

2. Geometry

We will consider a one-dimensional non-relativistic closed thin rod in three-dimensional flat space $\mathbb{R}^3$; its centre axis is a space curve C. We express a point along the curve C in $\mathbb{R}^3$ in terms of a vector $\boldsymbol{r}(q^1)$, where q^1 is its arclength. The length of the rod l is sufficiently large. We have an orthonormal system along C $(\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3)$ with fixing $\boldsymbol{n}_1$ as the tangent unit vector; $\boldsymbol{n}_1 = \partial_1 \boldsymbol{r}$, where $\partial_1 := \partial/\partial q^1$. We first make them satisfy the Frenet–Serre relation (Guggenheimer 1963),

$$\partial_1 \begin{pmatrix} \boldsymbol{n}_1 \\ \boldsymbol{n}_2 \\ \boldsymbol{n}_3 \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{n}_1 \\ \boldsymbol{n}_2 \\ \boldsymbol{n}_3 \end{pmatrix}. \tag{2.1}$$

Here $k = \partial_1 \phi$ is the curvature, $\tau = \partial_1 \theta$ is the Frenet–Serret torsion and they are functions of only q^1. We rotate the orthonormal frame SO(2) fixing $\boldsymbol{a}_1 := \boldsymbol{n}_1$ so that we obtain $(\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3)$ (da Costa 1981, Matsutani 1994b),

$$\partial_1 \begin{pmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \boldsymbol{a}_3 \end{pmatrix} = \begin{pmatrix} 0 & \kappa_1 & \kappa_2 \\ -\kappa_1 & 0 & 0 \\ -\kappa_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \boldsymbol{a}_3 \end{pmatrix} \tag{2.2}$$

where $\kappa_1 := k \cos\theta$, $\kappa_2 := k \sin\theta$ and

$$\theta := \int^{q^1} \tau \, \mathrm{d}q^1. \tag{2.3}$$

For convenience, we define the complex curvature as

$$v_c := v_1 + \mathrm{i}v_2 = \tfrac{1}{2}(\kappa_1 + \mathrm{i}\kappa_2) = \tfrac{1}{2}k\mathrm{e}^{\mathrm{i}\theta}. \tag{2.4}$$

If the rod is an elastica without elastic torsion, it obeys (Hashimoto 1972, Tsuru 1987)

$$\mathrm{i}\partial_t v_c - \tfrac{1}{2}\partial_1^2 v_c + 2|v_c|^2 v_c = 0. \tag{2.5}$$

Here we use the terminology 'elastic torsion' (Tsuru 1987). The elastic torsion can exist even if there is no Frenet–Serret torsion. Even for a straight line, the rod can twist and give elastic torsion. However in (2.5) the elastic torsion was neglected.

3. Dirac field on a rod in $\mathbb{R}^3$

Let us construct a Dirac equation in a thin rod (Matsutani and Tsuru 1992, Matsutani 1994a, b). If one prefers a moving elastic rod, our theory can easily be extended to a dynamic thin elastic rod after we set up the theory of the Dirac field in a static rod (Matsutani 1994b). Thus for the sake of simplicity, we will only deal with a static thin rod in this article.

The position $\boldsymbol{x} = (x^1, x^2, x^3)$ deviating from C in $\mathbb{R}^3$ written in Cartesian coordinates, can be expressed in terms of a curved coordinated system:

$$\boldsymbol{x} = \boldsymbol{r} + \boldsymbol{a}_2 q^2 + \boldsymbol{a}_3 q^3. \tag{3.1}$$

Here the Latin indices $(x^i, x^j, \ldots)$ indicate Cartesian coordinates and the Greek indices $(q^\mu, q^\nu, \ldots)$ curved coordinates along C. We have a tetrad in the vicinity of the rod $\zeta^i{}_\mu := \partial_\mu x^i$:

$$\zeta^i{}_\mu = (1 - h\delta_{\mu 1})a^i{}_\mu \qquad \text{not summed over } \mu \tag{3.2}$$

where $\partial_\mu := \partial/\partial q^\mu$, and

$$h(q^\mu) := \kappa_2(q^1)q^2 + \kappa_3(q^1)q^3. \tag{3.3}$$

Next, we extend $\mathbb{R}^3$ space to $(3+1)$-dimensional spacetime. Let us use Euclidean fermionic field theory. Let μ run from 0 to 3; the imaginary time of the fermionic system is $q^0 \equiv x^0$, $\zeta^0{}_\mu := \delta^0{}_\mu$ and $\zeta^i{}_0 := \delta^i{}_0$. The original Lagrangian density is given by

$$\mathcal{L}^{(3+1)\mathrm{D}} = \mathrm{i}\bar{\Psi}(\gamma^i\partial_i - m_0 - V)\Psi \tag{3.4}$$

where the $(3+1)$-dimensional γ-matrix γ^i is a 4×4 constant matrix, $\partial_i := \partial/\partial x^i$, m_0 is the bare mass and V is a confinement potential, $V := ((q^2)^2 + (q^3)^2)/2\delta$ with $\delta \to 0$. The potential V is not coupled with γ^0 and behaves like a mass so that we can avoid the Klein paradox on the confinement (Matsutani 1994a, for example Greiner 1990). (It is known that the Dirac particle cannot be confined in terms of the potential coupled with γ^0 in a region smaller than the Compton wavelength because the particle is exchanged with the antiparticle at the barrier owing to the distortion of the Dirac sea. However, since we effectively change its mass now, the Compton wavelength can be extremely small and, due to the mass-like potential, the particle cannot interact directly with the antiparticle.) In terms of the curved system, the Lagrangian density can be expressed by

$$\mathcal{L}^{(3+1)\mathrm{D}} = \zeta\bar{\Psi}\mathrm{i}(\gamma^\mu\partial_\mu - m_0 - V)\Psi + \mathcal{O}(q^2, q^3) \tag{3.5}$$

where $\gamma^\mu := \gamma^i\zeta_i{}^\mu$, and $\zeta := \det(\zeta^i_\mu) = 1 - h$. Because of the measure, $(1-h)\,\mathrm{d}^3q$, in the action integral, the space derivative $\mathrm{i}\partial_\mu$ cannot become Hermite. In order to avoid the difficulty, we redefine the field as $\psi^{(3+1)\mathrm{D}} = (1-h)^{1/2}\Psi$ (Matsutani 1994a).

Let us take the squeezing limit $\delta \to 0$. Here (q^2, q^3) vanishes and h approaches 1. The Lagrangian density along the rod becomes

$$\mathcal{L}[\bar{\psi}, \psi, v_1, v_2] = \bar{\psi}\mathrm{i}(\not{D} + m)\psi \tag{3.6}$$

where $\psi(q^0, q^1) := \psi^{(n+1)\mathrm{D}}(q^\mu)|_{q^2\equiv 0, q^3\equiv 0}$, m is a renormalized mass with the ground energy of V and

$$\not{D} := \gamma^0\partial_0 + \gamma^1\partial_1 + \tfrac{1}{2}\gamma^1\kappa_1 + \tfrac{1}{2}\gamma^2\kappa_2. \tag{3.7}$$

After taking the confinement limit, the γ-matrix γ^μ becomes independent of q^μ.

In order to simplify the argument, we will take the massless limit from now on. We express

$$\psi = \frac{1}{\sqrt{\delta q_0^0}} \sum_E \mathrm{e}^{\mathrm{i}Eq^0}\psi_E(q^1).$$

Here δq_0^0 is the integral region of the q^0-direction and is sufficiently large. The Dirac equation becomes (Matsutani 1994b)

$$E\psi_E = \mathrm{i}(\gamma^0\gamma^1\partial_1 + \gamma^0\gamma^2 v_1 + \gamma^0\gamma^3 v_2)\psi_E \tag{3.8}$$

where $v_\alpha := \kappa_\alpha/2$. In (3.8), we choose the expression for the γ-matrices in $(3+1)$-dimensional spacetime to be

$$\gamma^0 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad \gamma^1 = \begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix} \qquad \gamma^2 = \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix} \qquad \gamma^3 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}. \tag{3.9}$$

For $(\psi_E) := (\psi_{E+}^T, \psi_{E-}^T)^T$,

$$E\psi_{E+} = \mathrm{i}(\sigma^3\partial_1 + \sigma^1 \tfrac{1}{2}k\mathrm{e}^{\mathrm{i}\theta\sigma^3})\psi_{E+}. \tag{3.10}$$

Equation (3.10) has the same form as the Lax eigenvalue equation of the NLSE, i.e. the HVS (Hashimoto 1972, for example Drazin and Johnson 1989). Thus if we deal with an elastic rod, our quantum system is isomorphic to the quantum mechanics of the soliton (Matsutani 1994b).

In terms of the confinement prescription, we obtain the $(1+1)$-dimensional partition function (Matsutani 1994a)

$$Z[v_1, v_2] = \int \mathrm{D}\bar{\psi}\,\mathrm{D}\psi \exp\left(-\int \mathrm{d}^2 q\, \mathcal{L}[\bar{\psi}, \psi, v_1, v_2]\right). \tag{3.11}$$

4. Gauge transformation

We will consider the gauge transformation in our system. In the elastica problem, the shape of the rod makes the parity break down. Thus the gauge transformation in our problem is associated with the parity transformation (Matsutani 1994a).

Let us perturb the curvature v's around themselves for space direction (Matsutani 1994a). For an infinitesimal transformation,

$$v_\beta \mapsto v'_\beta := v_\beta(q^1) - \partial_1\alpha_\beta(q^1) \tag{4.1}$$

the Lagrangian changes as

$$\mathcal{L}[\bar{\psi}, \psi, v'_1, v'_2] = \mathcal{L}[\bar{\psi}, \psi, v_1, v_2] - \mathrm{i}\partial_1\alpha_1\bar{\psi}\gamma^2\psi - \mathrm{i}\partial_1\alpha_2\bar{\psi}\gamma^3\psi. \tag{4.2}$$

This extra term is cancelled by the gauge transformation

$$\psi \mapsto \psi' = \mathrm{e}^{\mathrm{i}(\gamma^1\gamma^2\alpha_1 + \gamma^1\gamma^3\alpha_2)}\psi \qquad \bar{\psi} \mapsto \bar{\psi}' = \bar{\psi}\mathrm{e}^{\mathrm{i}(\gamma^1\gamma^2\alpha_1 + \gamma^1\gamma^3\alpha_2)\mathfrak{T}} \tag{4.3}$$

if α's satisfy the relation

$$\alpha_1 = \frac{v_1}{v_2}\alpha_2. \tag{4.4}$$

Here $\mathfrak{T}$ is defined by

$$\mathfrak{T} : \mathcal{D}_0 \mapsto -\mathcal{D}_0 \tag{4.5}$$

and we impose the condition that it acts only on the differential operator. The transformation (4.5) has no physical meaning in relativistic manifold physics. However, in non-relativistic submanifold physics the embedding breaks the parity in the system locally and (4.5) is regarded as a physical transformation (see the appendix).

We rewrite the α's as

$$\alpha_1 =: v_1\alpha \qquad \alpha_2 = v_2\alpha. \tag{4.6}$$

Let us follow Fujikawa's prescription for an anomaly (Fujikawa 1979, Balachandran *et al* 1982, Matsutani 1994a). On the quantum level, the partition function changes as follows:

$$\begin{aligned} Z[v_1', v_2'] &= \int \mathrm{D}\bar{\psi}\, \mathrm{D}\psi \exp\left(- \int \mathrm{d}^2 q\, \mathcal{L}[\bar{\psi}, \psi, v_1', v_2'] \right) =: Z_1 \\ &= \int \mathrm{D}\bar{\psi}\, \mathrm{D}\psi \frac{\delta\psi\delta\bar{\psi}}{\delta\psi'\delta\bar{\psi}'} \exp\left(- \int \mathrm{d}^2 q\, \mathcal{L}[\bar{\psi}, \psi, v_1, v_2] \right) =: Z_2. \end{aligned} \tag{4.7}$$

Due to the natural relation between the left and right derivatives on the Dirac operator (Matsutani 1994a), we can expand the field by a complete set:

$$\psi = \sum_n a_n \varphi_n \qquad \bar{\psi} = \sum_n \bar{b}_n \chi_n^\dagger \tag{4.8}$$

satisfying

$$\mathrm{i}\mathcal{D}\varphi_n = \lambda_n \varphi_n \qquad \chi_n^\dagger \mathrm{i} \overleftarrow{\mathcal{D}} = -\lambda_n \chi_n^\dagger \tag{4.9}$$

with normalization

$$\int \chi_n^\dagger(q)\varphi_m(q)\, \mathrm{d}^2 q = \delta_{n,m}. \tag{4.10}$$

The fermionic measure is expressed by

$$\mathrm{D}\psi\, \mathrm{D}\bar{\psi} = \prod_m \mathrm{d}a_m\, \mathrm{d}\bar{b}_m. \tag{4.11}$$

Then transformation (4.3) becomes

$$\psi'(q) =: \sum_m a_m' \varphi_m(q) = \sum_m \mathrm{e}^{\mathrm{i}\alpha(\gamma^1\gamma^2 v_1 + \gamma^1\gamma^3 v_2)} a_m \varphi_m. \tag{4.12}$$

Let us cast the fermionic Jacobian in the transformations

$$\begin{aligned} a_m' &= \sum_n \int \mathrm{d}^2 q\, \chi_m^\dagger(q) \mathrm{e}^{\mathrm{i}\alpha(q)(\gamma^1\gamma^2 v_1 + \gamma^1\gamma^3 v_2)} \varphi_n(q) a_n \\ &=: \sum_n C_{m,n} a_n. \end{aligned} \tag{4.13}$$

Since they are Grassmannian variables, the Jacobian $(\delta\psi\delta\bar{\psi})/(\delta\psi'\delta\bar{\psi}')$ is expressed by (Fujikawa 1979)

$$\prod_m \mathrm{d}a'_m = [\det(\mathcal{C}_{m,n})]^{-1} \prod_m \mathrm{d}a_m \tag{4.14}$$

where

$$\begin{aligned}[\det(\mathcal{C}_{m,n})]^{-1} &= \left[\det\left(\delta_{m,n} + \mathrm{i}\int \mathrm{d}^2 q\, \alpha(q^1)\chi_m^\dagger(q)(\gamma^1\gamma^2 v_1 + \gamma^1\gamma^3 v_2)\phi_n(q)\right)\right] \\ &= \exp\left[-\mathrm{i}\sum_m \int \mathrm{d}^2 q\, \alpha(q^1)\chi_m^\dagger(q)(\gamma^1\gamma^2 v_1 + \gamma^1\gamma^3 v_2)\phi_m(q)\right] \\ &=: \exp\left[-\mathrm{i}\int \mathrm{d}^2 q\, \alpha(q^1)\mathcal{A}(q)\right].\end{aligned} \tag{4.15}$$

Here $\mathcal{A}(q)$ is ill defined and we must therefore regularize it. We adopt the heat kernel regularization procedure here (Fujikawa 1979, Balachandran *et al* 1982). The heat kernel is defined as (Gilkey 1984)

$$K(q, r, \tau) = \sum_m \mathrm{e}^{-\lambda_m^2\tau}\phi_m(q)\chi_m^\dagger(r) \tag{4.16}$$

and we redefine $\mathcal{A}(q)$,

$$\mathcal{A}(q) \equiv \lim_{\tau\to 0}\lim_{r\to q} \mathrm{tr}(\gamma^1\gamma^2 v_1 + \gamma^1\gamma^3 v_2)K(q, r, \tau). \tag{4.17}$$

For small τ, we can expand K asymptotically (Gilkey 1984)

$$K(q, r, \tau) \sim \frac{1}{4\pi\tau}\mathrm{e}^{-(q-r)^2/4\tau}\sum_{n=0}^{\infty} e_n(q, r)\tau^n. \tag{4.18}$$

The coefficient of the expansion is written by

$$e_1 = -\mathrm{i}(\gamma^1\gamma^2\partial_1 v_1 + \gamma^1\gamma^3\partial_1 v_2) + v_1^2 + v_2^2. \tag{4.19}$$

The trace over the spin space generates the factor 4 and we have

$$\begin{aligned}\mathcal{A}(q) &= -\mathrm{i}\frac{1}{4\pi}\,\mathrm{tr}(\gamma^1\gamma^2 v_1 + \gamma^1\gamma^3 v_2)(\gamma^1\gamma^2\partial_1 v_1 + \gamma^1\gamma^3\partial_1 v_2) \\ &= -\mathrm{i}\frac{1}{\pi}(v_1\partial_1 v_1 + v_2\partial_1 v_2).\end{aligned} \tag{4.20}$$

Noting that neither does $\mathfrak{T}$ have any effect on this procedure, we have the same term from the b_m's and we obtain the Jacobian

$$\frac{\delta\psi\delta\bar{\psi}}{\delta\psi'\delta\bar{\psi}'} = \exp\left[-2\mathrm{i}\int \mathrm{d}^2 q\, \alpha(q^1)\mathcal{A}(q)\right]. \tag{4.21}$$

Accordingly we evaluate the variation of the partition function (4.7) under gauge transformation (4.1).

5. Anomaly and index theorem on a submanifold

Let us derive the boson–fermion correspondence. The Ward–Takahashi identity in (4.7)

$$\frac{\delta}{\delta\alpha(q^1)}(Z_1 - Z_2)\bigg|_{\alpha(q^1)=0} \equiv 0 \tag{5.1}$$

gives an anomaly

$$\frac{1}{\delta q_0^0}\sum_E(\langle v_1\partial_1(\bar{\psi}_E\gamma^2\psi_E)\rangle + \langle v_2\partial_1(\bar{\psi}_E\gamma^3\psi_E)\rangle) = -\mathrm{i}\frac{2}{\pi}(v_1\partial_1 v_1 + v_2\partial_1 v_2). \tag{5.2}$$

Recalling (2.4), we obtain

$$\frac{1}{\delta q_0^0}\sum_E(\partial_1'\langle\bar{\psi}_E\gamma^2 \mathrm{e}^{\gamma^2\gamma^3\theta}\psi_E\rangle) = -\mathrm{i}\frac{1}{\pi}\partial_1|v_c| \tag{5.3}$$

where ∂_1' does not act on θ. If we define $\Gamma^2 := \gamma^2\mathrm{e}^{\gamma^2\gamma^3\theta}$, it seems to be constant in q^1. Noting the periodicity of the fermionic fields, we define

$$\begin{aligned} J_+ &:= \frac{\mathrm{i}}{2\pi}\int^{q^1}\mathrm{d}q^1\,\partial_1'\int\mathrm{d}E\,\psi_{E-}^*\sigma^1\mathrm{e}^{\mathrm{i}\sigma^3\theta}\psi_{E+} \\ J_- &:= \frac{\mathrm{i}}{2\pi}\int^{q^1}\mathrm{d}q^1\,\partial_1'\int\mathrm{d}E\,\psi_{E+}^*\sigma^1\mathrm{e}^{\mathrm{i}\sigma^3\theta}\psi_{E-}. \end{aligned} \tag{5.4}$$

Choosing an appropriate initial point q_0^1, we integrate (5.3) over q^1 and obtain

$$\langle J_+ - J_-\rangle = \frac{\mathrm{i}}{2\pi}k = \frac{1}{2\pi}\partial_1\phi. \tag{5.5}$$

The right-hand side is regarded as the absolute value of the NLSE soliton if the rod is an elastica. If we integrate it again, we obtain the global properties of the system:

$$\int\mathrm{d}q^1\,\langle J_+ - J_-\rangle = \frac{1}{2\pi}(\phi(l) - \phi(0)). \tag{5.6}$$

Since the rod we are considering is closed, the right-hand side gives an integer. It is worthwhile noting that (5.5) and (5.6) do not depend upon the starting point q_0^1 because of the periodicity at q_0^1 in (5.5). If we define the projection

$$\Pi : C \to C' \qquad \text{with } \Pi : \theta(q^1) \mapsto 0 \tag{5.7}$$

the right-hand side of (5.6) indicates the sum of the signed intersection number of the curve C'. We chose θ at q_0^1 as its origin, i.e. $\theta(q_0^1) = 0$.

Here we will investigate the geometrical meaning of the right-hand side of (5.6) $\mu := (\phi(l) - \phi(0))/(2\pi)$ (Matsutani 1994a). The μ seems to be associated with the winding number around a circle S^1, i.e. the fundamental group $\pi_1(S^1)$ (Nakahara 1990). Let us show how this is realized geometrically in terms of a closed loop.

In knot theory, one sometimes deals with a link diagram instead of the knot itself; the link diagram K_G is given through the projection of a (knotted or unknotted) loop K

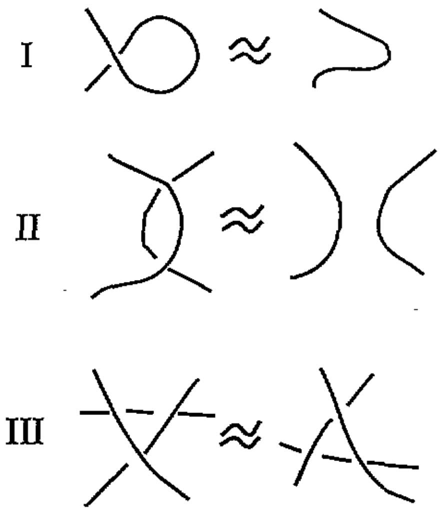

Figure 1. Reidemeister moves. From the classification of a knot, we divide the set of link diagrams by the equivalent class defined by the Reidemeister moves.

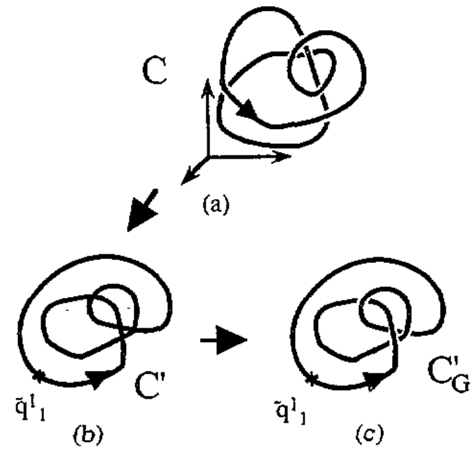

Figure 2. For a space curve C in (a), Π maps from C in (a) to C' in (b). In terms of C', we define the diagram C'_G.

in $\mathbb{R}^3$ to one in $\mathbb{R}^2$ (for example Kauffman 1988). K_G consists of crossings and curved segments. For a link diagram defined over $\mathbb{R}^2$, we can use the Reidemeister moves (figure 1). Equation (5.7) is the projection of a loop in $\mathbb{R}^3$ to one on $\mathbb{R}^2$. Thus we could use the result from knot theory.

We introduce a diagram C'_G associated with C. Due to (5.7), we define a diagram C', consisting of vertices and segments (figure 2). We affix an arrow along C' so that its direction corresponds to the orientation of the coordinate q^1 in C and choose a starting point $\tilde{q}_1^1$ along C'. We trace the curve along the arrow of C' from starting point $\tilde{q}_1^1$. When we first encounter each vertex, we draw a connected curve over it so that the curve that we are drawing now appears to be on another that we will draw. Accordingly on the second encounter with the vertex, we divide the segment into two pieces. When the drawing returns to the starting point, we obtain a new diagram, that is, C'_G (figure 2(c)). Due to the definition, C'_G is an unknotted link diagram. For diagram C'_G, we will introduce the 'winding' number. We assign the integer for the diagram

$$w(\text{[positive crossing]}) = 1 \qquad \text{and} \qquad w(\text{[negative crossing]}) = -1 \tag{5.8}$$

and

$$w(\text{[counterclockwise circle]}) = 1 \qquad \text{and} \qquad w(\text{[clockwise circle]}) = -1. \tag{5.9}$$

Summation (5.8) over the crossings is the same as the writhe number (Kauffman 1988).

When we refer to a part of the diagram C'_G to G_0, we will evaluate the 'winding' number on the Reidemeister moves (figure 1):

$$\begin{aligned}
w(G_0 + \text{[kink]}) &= -1 + w(G_0 + \text{[arc]}) \\
w(G_0 + \text{[kink]}) &= 1 + w(G_0 + \text{[arc]}) \\
w(G_0 + \text{[II]}) &= w(G_0 + \text{[two arcs]}) \\
w(G_0 + \text{[III]}) &= w(G_0 + \text{[III]}).
\end{aligned} \tag{5.10}$$

Accordingly $w(C'_G(\bar{q}_1^1))$ does not depend upon the starting point $\tilde{q}_1^1$. Furthermore it is clear that μ is in agreement with $w(C'_G)$. As we noted below (5.6), $\mu = w(C'_G)$ does not depend on the initial point q_0^1 and hence we can uniquely define the 'winding' number for a space curve C, $w(C) := w(C'_G)$. We, therefore, geometrically construct μ as

$$\mu = w(C). \tag{5.11}$$

Hence the right-hand side of (5.6) μ is the geometrical object.

The gauge field v is excited by a short-range force along the rod if the rod is an elastica. Thus it could not carry the long-range information as the long-range force, e.g. the electromagnetic force, gives the knot invariant as the linking number or the mutual induction (Witten 1990); the long-range force traverses the shortest course between the segments of the rod in $\mathbb{R}^3$ and then the segments interact and exchange their long-range information. The geometrical index μ is given by the integration of the local quantity and thus it is not directly connected with the knot invariant. However, for any knotted or unknotted closed loops, we can geometrically derive μ using the above scheme. Hence one should regard μ as a geometrical index related to the homotopy group $\pi_1(S^1)$. Furthermore the real DNA sometimes experiences a topological change, e.g. the transition from a knotted to an unknotted state. Before and after such a transition $w(C)$ might be conserved. We thus conclude that the index μ has geometrical meaning.

Since the geometrical index on the right-hand side is given through the projection Π (5.7), the left-hand side of (5.6) is also well defined under the projection. In fact in (5.3), if we make θ vanish, we can replace ∂'_1 with the ordinary partial differential ∂_1. In other words, our analytic index, the left-hand side of (5.6), is related to the algebraic structure of the Dirac operator $\mathcal{D}_{2D}$ in a rod on $\mathbb{R}^2$; the structure of $\mathcal{D}_{2D}$ is embedded in that of $\mathcal{D}$. As we show in the appendix, the analytic part is well defined after the projection and is guaranteed by the inverse scattering method on the MKdV equation (Drazin and Johnson 1989). In such considerations, the left-hand side is a type of difference between the right and left moving fields. Accordingly the left-hand side in (5.6) should be regarded as an analytic index.

Therefore (5.6) can be considered as the new index theorem related to the geometry of the submanifold.

Finally we will comment on the assumptions we have employed. Even though we dealt with the Dirac equation, our theory is non-relativistic. There is no contradiction arising from the thickness problem in relativistic theory. We should also note that the time direction in our theory does not play an important role because we add the flat $\mathbb{R}^1$ over the rod as the time; due to the Poincaré lemma, it can shrink to its origin homotopically and we need not consider the geometry for the direction. In fact, (5.6) does not explicitly depend upon time direction. Furthermore although we assumed that the length of the rod is sufficiently large, it is not difficult to develop the theory with a finite length l.

We thus conclude that our index theorem has geometrical meaning.

6. Conclusion

In this article, we have found the anomaly related to the submanifold system and the submanifold index theorem even though it is for a one-dimensional submanifold in $\mathbb{R}^3$.

Thus we summarize our index theorem as follows:

Theorem. (i) For a closed space curve $C \in \mathbb{R}^3$, we can draw the new curve C' so that the Frenet–Serre torsion of C ($\tau = \partial_s\theta$) vanishes over all C. The 'winding' number can be defined over C', $w(C)$ and is the geometrical index related to the homotopy group $\pi_1(S^1)$:

$$\mu = w(C). \tag{6.1}$$

(ii) We restrict the Dirac operator defined over the tubular neighbourhood of C and the time $\mathbb{R}^1$ to that along $C \times \mathbb{R}^1$ and obtain the Dirac operator $\not{D} = \gamma^\alpha D_\alpha$.

$$\not{D} = \begin{pmatrix} 0 & -\mathrm{i}\partial_0 + \sigma^3\partial_1 + \sigma_1 v \mathrm{e}^{\mathrm{i}\sigma^3\theta} \\ \mathrm{i}\partial_0 + \sigma^3\partial_1 + \sigma_1 v \mathrm{e}^{\mathrm{i}\sigma^3\theta} & 0 \end{pmatrix}. \tag{6.2}$$

We define the analytic index related to the map $\not{D}$ as

$$\nu := \int \mathrm{d}q^1 \, (J_+ - J_-) \tag{6.3}$$

where

$$\begin{aligned} J_+ &:= \frac{\mathrm{i}}{2\pi} \int^{q^1} \mathrm{d}q^1 \, \partial_1' \int \mathrm{d}E \, \psi_{E-}^* \sigma^1 \mathrm{e}^{\mathrm{i}\sigma^3\theta} \psi_{E+} \\ J_- &:= \frac{\mathrm{i}}{2\pi} \int^{q^1} \mathrm{d}q^1 \, \partial_1' \int \mathrm{d}E \, \psi_{E+}^* \sigma^1 \mathrm{e}^{\mathrm{i}\sigma^3\theta} \psi_{E-}. \end{aligned} \tag{6.4}$$

(iii) Both indexes μ and ν agree.

Anomaly (5.5) is given as the local version of the above theorem. In the theorem, we use the fact that the time direction can be regarded as the auxiliary axis in our theory.

As we have formulated the anomaly for a rod in $\mathbb{R}^3$, we can easily generalize this to a rod in $\mathbb{R}^n$ for $n > 3$ (Matsutani 1994b).

In our generalization from the Dirac operator on $\mathbb{R}^2$ to that in $\mathbb{R}^3$, we could not express the elastic torsion of the rod (Tsuru 1987). However, following the argument of Takagi and Tanzawa (1992), we may deal with it in terms of fermionic theory. In their theory for a Schrödinger particle instead of a Dirac particle, we can deal with its angular momentum around the rod. Thus we might express the twist of the rod in terms of their theory.

Recently Burgess and Jensen (1993) generalized our Dirac system on a rod to one on a curved surface in the flat space $\mathbb{R}^3$. A question arises as to whether their Dirac system is connected with the index theorem for a higher-dimensional submanifold in $\mathbb{R}^n$.

After submitting this paper, the author found that Shi and Hearst (1994) had applied the NLSE to the configuration of DNA and discovered the various shapes of DNA. It is expected that a concrete calculation of the fermionic field over it will be performed.

Appendix

In this appendix, we will give the algebraic structure of the analytic index of the Dirac operator in a rod on $\mathbb{R}^2$ (Matsutani 1994a). Therefore the notation in this appendix may differ from that in the main sections, but one can naturally translate them.

The Dirac operator in a rod on $\mathbb{R}^2$ is obtained if we project $\mathcal{D}$ in $\mathbb{R}^3$ as $\theta = 0$:

$$\mathcal{D}_{2D} := \gamma^0\partial_0 + \gamma^1\partial_1 + \gamma^2 v. \tag{A1}$$

Since this is a special case of $\mathcal{D}$, we can use the arguments in this article. For the operator and $\Gamma := \mathrm{i}\mathfrak{T}\gamma^1\gamma^2$, we have the relations

$$\{\Gamma, \mathcal{D}_{2D}\} = \Gamma\mathcal{D}_{2D} + \mathcal{D}_{2D}\Gamma = 0 \qquad \text{and} \qquad \Gamma^2 = 1. \tag{A2}$$

Here Γ behaves like γ^5 in the chiral anomaly (Fujikawa 1979). However it is related to the parity transformation (Alvarez-Gaumé, Pietra and Moore 1985) because for $(2+1)$-dimensional spacetime $\Gamma = \mathfrak{T}\gamma^0$ and the $(1+1)$-dimensional Dirac field confined in a rod on $\mathbb{R}^2$ still preserves the information of $(2+1)$-dimensional spacetime. Since our time direction q^0 has no geometrical structure and should be regarded as the auxiliary axis, the $\mathfrak{T}$ operator (4.5) has physical meaning. Then $P_\pm := (1 \pm \Gamma)$ is regarded as the projection operator. For the base functions (4.8) with $\theta = 0$, if we divide the function space $\{\phi_m\}$ into two parts $\{\phi_{\pm m}\}$ in terms of $P_\pm$, the set of non-zero modes $\{\phi_{+m}|\lambda_m \neq 0\}$ is bijective to $\{\phi_{-m}|\lambda_m \neq 0\}$ and this is true for $\{\chi_m\}$. Hence only the zero modes $\lambda_m = 0$ contribute an analytic index

$$\begin{aligned}\int \mathrm{d}q^2 \sum_m \chi_m^\dagger \Gamma\phi_m &= \int \mathrm{d}q^2 \Big(\sum_{+m} \chi_{+m}^\dagger \phi_{+m} - \sum_{-m} \chi_{-m}^\dagger \phi_{-m}\Big) \\ &= \frac{1}{2\pi}\int \mathrm{d}q^1\, \mathrm{d}E\, \partial_1(\bar{\psi}_E \gamma^2 \psi_E) =: \nu_0.\end{aligned} \tag{A3}$$

Let us introduce the trace of the operator in the function space associated with (4.17),

$$\mathcal{A}_{2D}(q,\tau) := \mathrm{Tr}\,\Gamma \mathrm{e}^{-(\mathrm{i}\mathcal{D}_{2D})^2\tau}. \tag{A4}$$

We can then easily prove that it does not depend on τ, $\partial_\tau \mathcal{A}_{2D}(q,\tau) = 0$ using (A2). Therefore we have the identity

$$\mathcal{A}_{2D}(q, \tau = \infty) = \mathcal{A}_{2D}(q, \tau = 0) \tag{A5}$$

and both sides give the indices along (A3) and (4.18) (Matsutani 1994a):

$$\mathcal{A}_{2D}(q, \tau = \infty) = \nu_0 \qquad \text{and} \qquad \mathcal{A}_{2D}(q, \tau = 0) = \frac{1}{2\pi}(\partial_1\phi(l) - \partial_1\phi(0)) =: \mu_0. \tag{A6}$$

Due to periodicity ν_0 vanishes. Thus for $\mathcal{D}_{2D}$, the anomaly corresponding to (5.3), which is a local version of (A3), is also determined by the zero modes $\lambda_m = 0$ and the index ν_0 is trivial, i.e. $\nu_0 = 0$.

However, we can introduce another index theorem such as (5.6). We define the indices as

$$\nu_{2D} := \int \mathrm{d}q^1\, \bar{\psi}_E^\dagger \gamma^2 \psi_E \qquad \mu_{2D} := \frac{1}{2\pi}(\phi(l) - \phi(0)) \tag{A7}$$

and obtain the index theorem related to the submanifold system as (Matsutani 1994a)

$$\nu_{2D} = \mu_{2D}. \tag{A8}$$

This is intrinsically the same as (5.6) and is also considered as its special case. Equation (A7) may be regarded as the transgression of (A6) (Gilkey 1984).

We can then find the geometrical meaning in the geometrical index μ_{2D} as we did in section 5.

On the other hand, the analytic index ν_{2D} is a type of difference between the right and left moving fields, which are given through the projection $P_{\pm}$. Since ν_{2D} is also given through the zero mode of (A1), it is determined in the frame of the classical field theory; $\lambda_n = 0$ gives the classical equation of motion of the field. Thus we can use the argument in the inverse scattering method of the MKdV (for example, Drazin and Johnson 1989), that is mathematically in the framework of classical field theory. In the inverse scattering method of soliton theory, we consider the map from the function space at $s = 0$ to that at $s = l$ and its inverse map; they are related to the scattering data. The kernel part of these maps correspond to the bounded state and the soliton solutions. The difference in dimension of both kernel parts, i.e. the index of the map, corresponds to the number of differences between the positive and negative MKdV solitons. The map (the inverse map) is related to maps between the right and left moving fields. Accordingly ν_{2D} is naturally understood and is regarded as the analytic index.

To the Dirac operator $\mathcal{D}$ in a rod in $\mathbb{R}^3$, we can apply the above argument. Then instead of Γ, the operator $\tilde{\Gamma} := \gamma^1\gamma^2 e^{\gamma^1\gamma^2\theta}$ play the same role in the system though $\{\tilde{\Gamma}, \mathcal{D}\} \neq 0$. Due to the property, $\partial_1^{\prime}$ appears in (5.3). Thus we should regard the algebraic structure generating to (5.6) as the pull-back of the projection to the structure of $\mathcal{D}_{2D}$. The projection is given through Π in (5.7) and the difficulty in the map comes from the fact that $[\Pi, \partial_1] \neq 0$. However, under such a meaning, the algebraic structure in our analytic system is well defined when it generates the analytic index (5.6).

References

Atiyah M F and Singer I M 1968a *Ann. Math.* **87** 484–530

—— 1968b *Ann. Math.* **87** 548–604

Alvarez-Gaume L, Pietra S D and Moore G 1985 *Ann. Phys.* **163** 288–317

Balachandran A P, Marmo G, Nair V P and Trahern C G 1982 *Phys. Rev.* D **25** 2713–16

Birrell N D and Davies P C 1982 *Quantum Fields in Curved Space* (Cambridge: Cambridge University Press)

Branson T P and Gilkey P B 1992 *Diff. Geom. Appl.* **2** 249–67

Burgess M and Jensen B 1993 *Phys. Rev.* A **48** 1861–6

da Costa R C T 1981 *Phys. Rev.* A **23** 1982–7

Drazin P G and Johnson R S 1989 *Solitons: An Introduction* (Cambridge: Cambridge University Press)

Fujikawa K 1979 *Phys. Rev. Lett.* **42** 1195–8

Greiner W 1990 *Relativistic Quantum Mechanics* (Berlin: Springer)

Gilkey P B 1984 *Invariance Theory, The Heat Equation, and the Atiyah–Singer Index Theorem* (Wilmington: Publish or Perish)

Guggenheimer H W 1963 *Differential Geometry* (New York: Dover)

Hashimoto H 1972 *J. Fluid Mech.* **51** 477–85

Ikegami M, Nagaoka Y, Takagi S and Tanzawa T 1992 *Prog. Theor. Phys.* **88** 229–49

Jensen H and Koppe H 1971 *Ann. Phys.* **63** 586–91

Kauffman L H 1988 *Contemp. Math.* **78** 263–97

Matsutani S 1993a *Phys. Rev.* A **47** 686–9

—— 1993b *J. Phys. A: Math. Gen.* **26** 5133–43

—— 1994a *Prog. Theor. Phys.* **91** 1005–37

—— 1994b *Phys. Lett.* **189A** 27–31

Matsutani S and Tsuru H 1991 *Phys. Rev.* A **46** 1144–7

Nakahara M 1990 *Geometry, Topology and Physics* (Bristol: Adam Hilger)

Schlick T and Olson W K 1992 *Science* **257** 1110–15

Shaw S Y and Wang J C 1993 *Science* **260** 533–6
Shi Y and Hearst J E 1994 *J. Chem. Phys.* **101** 5186–200
Takagi S and Tanzawa T 1992 *Prog. Theor. Phys.* **87** 937–42
Tokihiro T and Hanamura E 1993 *Phys. Rev. Lett.* **30** 1423–6
Tsuru H 1987 *J. Phys. Soc. Japan* **56** 2309–24
Tsuru H and M Wadati 1986 *Biopolymers* **25** 2083–96
Witten E 1989 *Commun. Math. Phys.* **111** 613–65